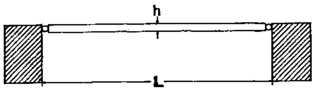

Fig. 1 Plate strip

The problem now will be considered in more detail. In Ref. 2, for instance, it is stated (in connection with a circular plate problem) that "For thin plates the temperature may be approximated by $T_0(r,\theta,t) + zT_1(r,\theta,t)$ where the effects of the two terms may be superimposed if the deflections are small and the material is linearly elastic." Newman and Forray place no restriction on the magnitude of T_0 or the corresponding forces caused by this term. The assumption of inequality (2) is needed, however. An example now will be given in order to indicate how restrictive this inequality can be. Consider a "two-dimensional" plate strip for simplicity (Fig. 1). The plate is pinned between two immovable supports and is of length L, thickness h, and made of aluminum. It is loaded by a uniform cooling of $T_0°$ and by a system of thermal moments. For a plate 28 in. long and 0.100 in. thick which deflects in an approximate half-sine wave, it is found that the superposition principle will give an answer in error by 100% if T_0 is as large as 1°F. In other words, neglecting the midplane restoring forces due to a 1° temperature drop will cause the calculated deflection to be twice as large as the correct value. For an accurate answer, T_0 would have to be limited to much less than 1°F in this case! (It is realized that a temperature rise of only 1°F would cause this same plate to buckle. This is a different question, however, and is not of interest here.)

Of course, counter-examples can be constructed which would be less restrictive, particularly for thicker plates and for plates free to expand. For thin plates in general, however, it appears that the superposition is applicable only to cases where the midplane stresses are so low that they may as well be neglected in the determination of the maximum stresses. If this is the case, then the superposition is not needed.

References

[1] Newman, M. and Forray, M., "Thermal stresses and deflections in thin plates with temperature-dependent elastic moduli," J. Aerospace Sci. **29**, 372–373 (1962).

[2] Newman, M. and Forray, M., "Bending of circular plates due to asymmetric temperature distribution," J. Aerospace Sci. **28**, 773–778 (1961).

[3] Newman, M. and Forray, M., "Thermal stresses and deflections in rectangular panels, Part II," Aeronaut. Systems Div. TR 61-537, Part II (1962).

[4] Newman, M. and Forray, M., "Axisymmetric large deflections of circular plates subjected to thermal and mechanical loads," J. Aerospace Sci. **29**, 1060 (1962).

[5] Timoshenko, S., *Theory of Plates and Shells* (McGraw-Hill Book Co. Inc., New York, 1940), pp. 299–307.

Reply by Authors to William J. Anderson

MALCOLM NEWMAN* AND MARVIN FORRAY†
Republic Aviation Corporation, Farmingdale, N. Y.

ANDERSON states in his note that the inclusion of membrane restoring forces in the plate equilibrium equation is "generally considered a part of classical linear plate theory." This statement is debatable, since Eq. (1) cannot be derived from variational principles without the inclusion of nonlinear terms in the strain-displacement relations. Novozhilov[1] remarks that when these nonlinear terms are neglected "one obtains the formulas of the classical theory of plates." Furthermore, regardless of what the expression "classical plate theory" means, the present authors clearly stated, in the references cited by Anderson, that superposition of the bending and in-plane problems is being used. It is true that the limitations of this superposition technique have not been established. However, the determination of the range of applicability of this method often presents a very difficult task. In the class of problems investigated, the temperature and hence the membrane forces are permitted to vary over the plate planform. If the deflections are small, Anderson's requirement that the inequality (2) be satisfied *throughout* the plate is a sufficient but not necessary condition for the applicability of superposition. It is far too restrictive, since superposition still may be used when (2) is satisfied on some average basis rather than pointwise. For example, superposition should yield accurate solutions (with interior bending and membrane stresses of the *same* order of magnitude) for a wide variety of nonuniform heating problems in which the edges are unrestrained in the plane of the plate. These are very practical problems, since important structural components usually are designed to permit thermal expansion. As a guide to the reader, superposition generally can be considered valid when the absolute ratio of the generated edge thrust to the buckling thrust is small compared to unity.

No one will deny the inaccuracy of superposition for an axially restrained thin strip. The authors never intended, however, that the technique be used indiscriminately.

In closing, it must be remarked that Refs. 2–8 present solutions to many thermoelastic in-plane and bending problems. Anderson neglects to mention that either type of problem is of importance in itself.

Received April 8, 1963.

* Specialist Engineer, Structures.

† Development Engineer, Structures.

References

[1] Novozhilov, V. V., *Foundations of the Nonlinear Theory of Elasticity* (Graylock Press, Rochester, N. Y., 1953), p. 183.

[2] Newman, M. and Forray, M., "Bending stresses due to temperature in hollow circular plates," J. Aerospace Sci. **27**, 717–718 (1960).

[3] Forray, M. and Newman, N., "Bending of circular plates due to asymmetric temperature distribution," J. Aerospace Sci. **28**, 773–778 (1961).

[4] Switzky, H., Forray, M., and Newman, M., "Thermostructural analysis manual," Wright Air Dev. Div. TR-60-517, Vol. I (August 1962).

[5] Forray, M., Newman, M., and Kossar, J. "Thermal stresses and deflections in rectangular panels—Part I," Aeronaut. Systems Div. TR 61-537 (November 1961).

[6] Forray, M., Newman, M., and Switzky, H., "Thermostructural analysis manual, Vol. II," Aeronaut. Systems Div. TR-61-537 (February 1962).

[7] Newman, M. and Forray, M., "Thermal stresses and deflections in thin plates with temperature-dependent elastic moduli," J. Aerospace Sci. **29**, 372–373 (1962).

[8] Newman, M. and Forray, M., "Thermal stresses and deflections in rectangular panels—Part II, The analysis of rectangular panels with three dimensional heat inputs," Aeronaut. Systems Div. TR-61-537 (October 1962).

Comment on "Error Matrix for a Flight on a Circular Orbit"

HERMANN M. DUSEK*
General Motors Corporation, El Segundo, Calif.

IN a recent technical note,[1] Wisneski derives the well-known solutions of the linear perturbation differential

Received by ARS October 11, 1962; revision received October 29, 1962.

* Head, Space Studies, AC Spark Plug Division. Member AIAA.

equations in a Newtonian central-force field for a circular nominal trajectory. Following Eq. (5) he states that "Eqs. (5) are applicable to perturbations of a general elliptical trajectory. However, the integration to obtain δr and $\delta\theta$ becomes a difficult task because of dependence on time of the unperturbed parameters r and θ." It will be shown that the solution by elementary functions is possible if the eccentric anomaly E is used as an independent variable. The coordinate system is chosen in such a way that the z axis is perpendicular to the plane of the unperturbed motion. Then, the rigorous differential equation of motion in spatial cylindrical coordinates (r, θ, z) are

$$\ddot{r} + \frac{Kr}{(r^2 + z^2)^{3/2}} - r\dot{\theta}^2 = 0 \qquad \frac{d}{dt}(r^2\,\dot{\theta}) = 0 \tag{1}$$

and

$$\ddot{z} + \frac{Kz}{(r^2 + z^2)^{3/2}} = 0 \tag{2}$$

The solution corresponding to the nominal initial conditions is called $r = r_0$, $\theta = \theta_0$. $z_0 \equiv 0$ due to the choice of the coordinate system. The solution corresponding to the perturbed initial condition is denoted by

$$r = r_0 + \delta r \qquad \theta = \theta_0 + \delta\theta \qquad z = \delta z \tag{3}$$

In a straightforward manner, one now could derive equations linear in δr, $\delta\theta$, and δz. But it is immediately clear from the character of Eq. (1) that such a system of differential equations of δr and $\delta\theta$ would be coupled. To avoid this difficulty, the energy integral[2] is used:

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - \frac{2K}{(r^2+z^2)^{1/2}} + \frac{K}{a_0} = C_3 \tag{4}$$

where a_0 is the major axis of the unperturbed elliptical motion, K the gravitational constant of the central force field, and C_3 a constant of integration which vanishes if $r = r_0$ and $\theta = \theta_0$. Multiplying the differential equations for r in (1) by r, adding (4), and linearizing yields

$$[d^2(r_0\delta r)/dt^2] + (K/r_0^3)(r_0\,\delta r) = C_3 \tag{5}$$

On the other hand, one obtains by a twofold subtraction of Eq. (5) from the linearized energy integral

$$\frac{d\delta\theta}{dt} = \frac{1}{na_0^2(1-e^2)^{1/2}}\left[\frac{d}{dt}\left(2\,\frac{d(r_0\delta r)}{dt} - \delta r\,\frac{dr_0}{dt}\right) - \frac{3C_3}{2}\right] \tag{6}$$

Finally, direct linearization of Eq. (2) yields

$$[d^2(\delta z)/dt^2] + (K/r_0^3)\delta z = 0 \tag{7}$$

The integration problem is reduced to the solution of one differential equation of second order, namely

$$(d^2q/dt^2) + (K/r_0^3)q = C \tag{8}$$

With the well-known relation

$$\begin{aligned} ndt &= [1 - e\cos(E' + E_0)]dE' \\ n^2 &= K/a_0^3 \qquad E' = E - E_0 \end{aligned} \tag{9}$$

where e is the eccentricity, E the eccentric anomaly, and E_0 the eccentric anomaly corresponding to the beginning of the motion, one derives from the homogeneous part of Eq. (8)

$$[1 - e\cos(E' + E_0)](d^2q/dE'^2) - e\sin(E' + E_0)\,(dq/dE') + q = 0 \tag{10}$$

A fundamental solution of (10) with the Wronskian equal to one is

$$\begin{aligned} q_1 &= n^{-1/2}\,[\cos E' - e\cos E_0] \\ q_2 &= n^{-1/2}\,[\sin E' + e\sin E_0] \end{aligned} \tag{11}$$

Thus, the general solution of (8) can be written as

$$q = C_1q_1 + C_2q_2 + C\left\{q_2\int_0^{E'} q_1\,dt - q_1\int_0^{E'} q_2\,dt\right\}$$
$$E' = E - E_0 \tag{12}$$

From Eqs. (9) and (11) it follows that the integrals in (12) can be expressed as trigonometric functions of E and rational functions of e. For the sake of brevity, the explicit expressions are not written down, but it might be pointed out that the six constants of integration can be determined independently from each other if the quantity $E' = E - E_0$ is introduced. This follows easily from the character of Eqs. (6, 7, and 12). For an application of this classical method in perturbed central force fields, see Ref. 3.

References

[1] Wisneski, M. L., "Error matrix for a flight on a circular orbit," ARS J. **32**, 1416–1418 (1962).

[2] Brouwer, D. and Clemenc, G. M., *Methods of Celestial Mechanics* (Academic Press, New York and London, 1961).

[3] Dusek, H. M., "Theory of error propagation in astro-inertial guidance systems for low-thrust earth orbital missions," ARS Preprint 2683-62 (November 1962).

Reply to Comment by H. M. Dusek

M. L. Wisneski*
The Boeing Company, Seattle, Wash.

THE comment of Mr. Dusek and his ARS meeting paper are very interesting. It appears that a manageable solution of Eqs. (5) of his Ref. 1 can be obtained if the technique of his Ref. 2 for dealing with such equations is employed. The technique uses eccentric anomaly as an independent variable. However, to express perturbations in terms of the true anomaly rather than the eccentric anomaly, the conversion involves a series expansion. Thus an additional approximation, strictly valid for low-eccentricity orbits, has to be made when only the first few terms are retained.

In Dusek's Ref. 1 signs in two places are incorrect. Referring to the last two "error matrices," the signs in front of expressions in positions 2 and 3 should be plus.

Received by ARS November 26, 1962.

* Research Engineer, Physics Technology Department, Aero-Space Division.

Response to Author's Reply

Hermann M. Dusek*
General Motors Corporation, El Segundo, Calif.

THE introduction of the true anomaly is pertinent neither to my original comment nor to Wisneski's original paper. Wisneski states, "However, to express perturbations in terms of the true anomaly rather than the eccentric anomaly, the conversion involves a series expansion." This statement, in itself, certainly is true if one follows the classical astronomical practice (see Ref. 2, pp. 62–65, of the original comment). However, to this author's knowledge, series expansions for arbitrary eccentricities of the unperturbed orbit in connection with this particular problem can be avoided.

Received March 28, 1963.

* Head, Space Studies, AC Spark Plug Division. Member AIAA.